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Research Article

Dynamic Analysis of an Ecological Model with Fear Effect on Prey and Additional Food for Predator Using Ratio-Dependent Functional Response

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Abstract

This study discusses the dynamic analysis of a predator-prey model that incorporates the fear effect on prey and supplemental food for predators, using a ratio-dependent functional response. The fear effect reduces the prey's intrinsic growth rate due to behavioral changes under predation risk, while supplemental food enables the predator to survive even when prey density is low. The analysis begins with the formulation of the model equations, followed by the identification of equilibrium points, linearization of the system, and local stability analysis using eigenvalues. Numerical simulations are carried out using Matcont and Pplane to verify the analytical results and to illustrate the system's qualitative behavior. The model parameters are based on the interaction between elk (ELK) as prey and wolves (Canis lupus) as predators. The results reveal four equilibrium points: $E_0 = (0,0)$ *is an unstable nodal source,* $E_1 = \left(0, \dfrac{nA(c\gamma - e\alpha)}{em}\right)$ *and* $E_2 = \left(\dfrac{r-a}{b}, 0\right)$ *are unstable saddle points, while the coexistence equilibrium* $E_3 = (x^*, y^*)$ *is a stable spiral sink under certain parameter conditions. Bifurcation analysis with respect to the fear parameter* f *and the supplemental food parameter* A *reveals the occurrence of a transcritical bifurcation, where two equilibrium branches exchange stability. The system tends toward the equilibrium point* E_1 *when either* f *or* A *exceeds a critical threshold, indicating that predators can persist even as prey populations decline significantly. These findings suggest that predator survival is not solely dependent on prey availability but also influenced by the availability of alternative food sources and the intensity of the prey's fear response.*

Keywords : *Predator-prey · Fear effect · Additional food · Ratio-dependent Functional response · Transcritical bifurcation*
MSC2020 : *92D25 · 34D20 · 37N25 · 34C23*

1. Introduction

The interaction between predators and prey is a fundamental component of population dynamics and ecosystem stability. Mathematical models serve as powerful tools for researchers to explore how various ecological factors influence the growth and survival of both predators and prey over time.

One of the earliest and most influential models for studying predator–prey dynamics is the Lotka–Volterra model, introduced in the 1920s [1]. This model illustrates how predator populations depend on prey as a food source, while prey populations are regulated by predation pressure. Although foundational, the model omits several real-world complexities that significantly affect population dynamics in natural ecosystems.

To enhance ecological realism, various extensions have been proposed. One such refinement is the incorporation of the *fear effect*, where the mere presence of predators induces behavioral changes

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in prey—even in the absence of direct predation. Fear can lead to reduced foraging, lower energy intake, and decreased reproductive success [2], which in turn can indirectly influence predator populations [3–6]. These non-consumptive effects play a critical role in shaping community structure and trophic interactions.

Another key factor is the presence of *supplemental food* for predators. In both natural and managed ecosystems, predators may rely on alternative food sources when prey is scarce, thereby stabilizing predator populations and reducing pressure on the main prey species [3, 6, 7]. This dynamic enhances ecological resilience and can promote long-term coexistence.

The functional relationship between prey availability and predation rate—known as the *functional response*—is also central to understanding predator–prey dynamics. Introduced by Holling in the 1950s [8], this concept has since evolved into several more complex forms, including the Beddington–DeAngelis [9], Monod–Haldane [10], and Crowley–Martin [11] models. Among these, *ratio-dependent* models [4, 12] are particularly effective for capturing dynamics in systems where predator interference affects feeding efficiency. Unlike prey-dependent models, ratio-dependent models incorporate both prey and predator densities, making them more suitable for ecosystems involving group-hunting predators such as wolves (*Canis lupus*) or large herbivores like elk (*Cervus canadensis*).

Although Mondal et al. (2018) proposed a model that includes the Holling Type II functional response, fear effect, and supplemental food [3], their formulation remains prey-dependent and thus may not fully capture predator interference or adaptive foraging behavior observed in real ecosystems. To address this limitation, we propose a new predator–prey model that integrates a *ratio-dependent functional response*, while retaining the behavioral (fear) and environmental (supplemental food) components. This formulation offers a more ecologically realistic framework to investigate how behavioral adaptation and resource supplementation jointly influence coexistence, extinction thresholds, and system stability.

Recent theoretical and empirical studies have highlighted that fear-induced behavioral changes can cause cascading effects across trophic levels—even in the absence of direct predation [2, 13]. These *non-consumptive effects* are now recognized as critical to ecosystem function, affecting nutrient cycling, spatial distribution, and population resilience. Likewise, the presence of supplemental food—whether through natural availability or human intervention—can mitigate population collapse and promote predator–prey coexistence, especially in fragmented or anthropogenically altered habitats [14–16]. Integrating these ecological processes into mathematical models yields not only theoretical insights but also practical guidance for biodiversity conservation, wildlife management, and ecological resilience under growing anthropogenic pressures [17, 18].

This paper is organized as follows: Section 2 presents the mathematical formulation of the model. Section 3 provides the analytical results, including equilibrium analysis and bifurcation structure. Section 4 offers numerical simulations to support the theoretical findings. Finally, Section 5 concludes with a discussion of the ecological implications and future research directions.

2. Model Formulation

We start the model formulation by adapting the Lotka-Volterra equations given by

$$\begin{aligned}\frac{dx}{dt} &= rx - cxy\\ \frac{dy}{dt} &= \gamma xy - ey\end{aligned} \tag{1}$$

where x and y denote the population density of prey and predator. The parameter r, c, γ, and e respectively express the intrinsic growth rate of prey, predation rate of a predator on prey, conversion of predation rate to the birth rate of a predator, and the natural mortality rate of a predator.

Of the many ecological issues studied, interactions between two or more populations with predator-prey relationships have received increasing attention. New mathematical models are continually developed to better reflect real-world ecological conditions by incorporating various influencing factors. For example, the study in [7] investigated the dynamics of a Leslie-Gower predator-prey model with additional food provided for the predator. Furthermore, research in [12] examined the dynamics of a discrete-time Leslie-Gower model with a ratio-dependent response function, while [3] investigated the combined effects of fear and additional food in a delayed predator–prey model.

The Equation (1) becomes a basic model for researchers to develop new models based on the reality that occurs in nature. In this study, we analyzed the influence of two ecological factors, the fear effect, which reduces the prey's reproduction rate due to the presence of predators [3], and additional food, which can enhance predator growth and persistence [3, 7]. ssuming that the presence of predators can affect the birthrate of the prey population. Numerous studies have shown that the presence of predators can havean impact on prey, which is more important than direct predation for controlling population dynamics [2]. By applying the fear effect term to model (1), we obtain the following modified Lotka-Volterra model.

$$
\begin{aligned}
\frac{dx}{dt} &= \frac{rx}{1+fy} - cxy \\
\frac{dy}{dt} &= \gamma xy - ey
\end{aligned}
\tag{2}
$$

where f is fear level of the prey due to the appearance of the predator.

Beyond fear, another important factor is the availability of additional food sources for predators. In many natural and managed systems, predators may consume not only prey but also other sources of food, such as carrion, crops, or food provided artificially. This additional food can reduce the predator's dependence on prey, thereby altering the interaction dynamics. We assume that the predator receives a constant supply of external food, denoted by nA, where A is the amount of food per feeding site, and n is the number of such sites. Following the approach in [3] and [7], we modify the predator's growth term to account for both prey consumption and external food. The resulting term becomes $c\gamma(x+nA)y$, where $x+nA$ reflects the combined availability of prey and supplemental food.

However, at this stage, the model still employs a prey-dependent functional response, specifically the term cxy, which may not fully reflect ecological realism. In natural ecosystems, the efficiency of predation tends to decline as predator density increases due to factors such as mutual interference, territorial behavior, and intra-guild competition. These phenomena are not captured adequately by prey-dependent models, which assume that predation depends solely on prey abundance. To address this limitation, we incorporate a ratio-dependent functional response, which considers the ratio between prey and predator populations.

The ratio-dependent predation term used in this study is defined as:

$$
\frac{cxy}{my+\alpha nA+x}, \tag{3}
$$

where m is a half-saturation constant representing the influence of predator density, and α reflects the quality or effectiveness of additional food in contributing to predator growth.

This modification is motivated by both ecological evidence and mathematical advantages. As highlighted by Arditi and Ginzburg [12], ratio-dependent models account for the reduced efficiency of predation at higher predator densities and are more suitable for systems with significant predator interference or alternative food sources. Moreover, such models avoid the biologically unrealistic outcomes of prey-dependent formulations, such as unbounded predation rates under high prey availability. They also ensure boundedness in the system and promote more consistent population dynamics [3, 12].

The denominator in Equation (3) modulates predation efficiency by incorporating predator density (my) and the availability of external food (αnA), reflecting the idea that both crowding and alternative resources dilute predation intensity. Consequently, the predator's growth term is modified to:

$$\frac{c\gamma(x+nA)y}{my+\alpha nA+x}-ey, \tag{4}$$

which combines the effects of ratio-dependent consumption and the energetic benefit from supplemental food.

Unlike the classic term γxy, which assumes predator growth is entirely driven by prey consumption, this formulation accounts for the contribution of additional food sources. Following the approaches in [3, 7], we replace the prey-only term with $c\gamma(x+nA)y$, where $(x+nA)$ represents the combined intake from prey and additional food. This structure better captures ecological observations that external feeding reduces the predator's reliance on prey and can help stabilize predator-prey dynamics.

Moreover, it is essential to account for the environmental carrying capacity in the prey population. In natural ecosystems, prey growth is typically constrained by limited resources such as food, space, or shelter. To incorporate this ecological constraint, we introduce a logistic-type growth adjustment by adding density-dependent mortality terms to the prey equation. Specifically, the prey population is subject to a natural death rate ax, as well as an intraspecific competition term bx^2. These terms collectively form a logistic-regulated decrease in prey population:

$$ax-bx^2 \tag{5}$$

Combining all these ecological components—namely the fear effect, logistic growth, ratio-dependent predation, and additional food supply—we obtain the complete model as follows:

$$\begin{aligned}\frac{dx}{dt}&=\frac{rx}{1+fy}-ax-bx^2-\frac{cxy}{my+\alpha nA+x},\\ \frac{dy}{dt}&=\frac{c\gamma(x+nA)y}{my+\alpha nA+x}-ey.\end{aligned} \tag{6}$$

The parameters used in this study can be seen in the following Table 1.

Table 1. Biological interpretation of parameters

Parameter	Description
r	Intrinsic growth rate of prey population
f	Strength of fear effect on prey
a	Natural death rate of prey
b	Intraspecific competition rate among prey
c	Maximum predation rate of predator on prey
n	Quantity or frequency of additional food provided
A	Amount per unit of additional food
m	Prey density at half-maximum consumption rate
α	Quality effect of additional food
γ	Conversion efficiency of food into predator growth
e	Natural death rate of predator
x	Density of prey population
y	Density of predator population

The parameter values used in this study are based on assumptions and values referenced from the work of Mondal et al. [3], which investigates similar ecological dynamics involving fear effects and additional food in predator-prey models. Since exact empirical data are not always available, the parameters were selected to reflect biologically reasonable conditions as demonstrated in their study. This approach ensures the ecological relevance and plausibility of the model's structure and behavior.

3. Analytical Results

In this section, we analyze the dynamic behavior of the model (6) analytically, focusing on the existence and stability of equilibrium points. The equilibrium points of model (6) are determined by solving the following system:

$$\begin{aligned} x\left(\frac{r}{1+fy}-a-bx-\frac{cy}{my+\alpha nA+x}\right)&=0,\\ y\left(\frac{c\gamma(x+nA)}{my+\alpha nA+x}-e\right)&=0. \end{aligned} \tag{7}$$

The first equilibrium point is the origin $E_0=(0,0)$, which corresponds to the extinction of both prey and predator populations. The second equilibrium point, denoted as $E_1=\left(0,\frac{nA(c\gamma-e\alpha)}{em}\right)$, represents the predator-only equilibrium, where the predator persists due to the supplemental food source, even though the prey goes extinct. The third equilibrium point, $E_2=\left(\frac{r-a}{b},0\right)$, is the prey-only equilibrium, where the prey population stabilizes in the absence of predators.

The fourth equilibrium point is an interior equilibrium, denoted by $E_3=(x^*,y^*)$, where both populations coexist. Here,

$$x^*=\frac{emy^*+e\alpha nA-c\gamma nA}{c\gamma-e},$$

and y^* is the positive solution of the following quadratic equation:

$$ay^{*2}+by^*+c=0 \tag{8}$$

with the coefficients given by:

$$\begin{aligned} a&=f\left[m(-a-bx^*)-c\right],\\ b&=rm-am-af\alpha nA-c+x^*\left(-af-bm-bfx^*-bf\alpha nA\right),\\ c&=\alpha nA(r-a)+x^*\left(r-bx^*-a-b\alpha nA\right). \end{aligned}$$

The existence conditions of all equilibrium points in the system are given as follows in

Lemma 1. *The trivial equilibrium point $E_0=(0,0)$ of the system (6) always exists. The axial equilibrium point $E_1=\left(0,\frac{nA(c\gamma-e\alpha)}{em}\right)$ exists if $c\gamma>e\alpha$. The prey-only equilibrium point $E_2=\left(\frac{r-a}{b},0\right)$ exists if $r>a$. The interior equilibrium point $E_3=(x^*,y^*)$ exists if $c\gamma>e$ and $emy^*+e\alpha nA>c\gamma nA$, with x^* given by $x^*=\frac{emy^*+e\alpha nA-c\gamma nA}{c\gamma-e}$. Furthermore, the existence of a positive solution y^* requires that the discriminant $\Delta=b^2-4ac>0$ and at least one of the roots of the quadratic equation $ay^{*2}+by^*+c=0$ is positive.*

To analyze the local behavior of solutions near the equilibrium points, we linearize model (6). This process yields the following Jacobian matrix:

$$J=\begin{bmatrix} \frac{r}{1+fy}-a-2bx-\frac{cy(my+\alpha nA)}{(my+\alpha nA+x)^2} & -\frac{rfx}{(1+fy)^2}-\frac{cx(my+\alpha nA+x)-cmxy}{(my+\alpha nA+x)^2}\\ \frac{cy\gamma(my+\alpha nA+x)-cy\gamma(x+nA)}{(my+\alpha nA+x)^2} & \frac{c\gamma(x+nA)(\alpha nA+x)}{(my+\alpha nA+x)^2}-e \end{bmatrix} \tag{9}$$

The Jacobian matrix at the trivial equilibrium point $E_0 = (0,0)$ is given by

$$J_0 = \begin{bmatrix} r-a & 0 \\ 0 & c\gamma - e \end{bmatrix}.$$

The eigenvalues of J_0 are $\lambda_1 = r-a$ and $\lambda_2 = c\gamma - e$. Based on these eigenvalues, then we have the following theorem.

Theorem 2. *The trivial equilibrium point $E_0 = (0,0)$ is locally asymptotically stable if and only if $r < a$ and $c\gamma < e$.*

This result implies that if the prey's intrinsic growth rate is insufficient ($r < a$) and the predator cannot sustain itself due to low food availability or predation inefficiency ($c\gamma < e$), both species are driven to extinction over time.

The Jacobian matrix J_1 at E_1 is given by

$$J_1 = \begin{bmatrix} \dfrac{r}{1+\dfrac{fnA(c\gamma-e\alpha)}{em}} - a - \dfrac{c(c\gamma-e\alpha)}{mc\gamma} & 0 \\ \dfrac{(c\gamma-e\alpha)(c\gamma-e)}{mc\gamma} & \dfrac{\alpha e^2}{c\gamma} - e \end{bmatrix} \tag{10}$$

We observe that one eigenvalue is $\lambda_1 = \dfrac{r}{1+\frac{fnA(c\gamma-e\alpha)}{em}} - a - \dfrac{c(c\gamma-e\alpha)}{mc\gamma}$ and the other eigenvalue is $\lambda_2 = \dfrac{\alpha e^2}{c\gamma} - e$.

Based on these eigenvalues, then we have the following theorem.

Theorem 3. *The equilibrium point E_1 is locally asymptotically stable if and only if $\dfrac{rem}{em + fnA(c\gamma - e\alpha)} < a + \dfrac{c\gamma - e\alpha}{m\gamma}$ and $\dfrac{\alpha e}{c\gamma} < 1$.*

This equilibrium corresponds to a predator-only scenario. It is biologically feasible when the predator receives adequate additional food to survive without prey. The condition for stability indicates that predator persistence depends critically on the quantity and quality of supplemental food relative to its natural death rate.

The Jacobian matrix J_2 at E_2 is given by

$$J_2 = \begin{bmatrix} -r+a & -rf\dfrac{r-a}{b} - \dfrac{c \cdot \frac{r-a}{b}}{\alpha nA + \frac{r-a}{b}} \\ 0 & \dfrac{c\gamma\left(\frac{r-a}{b} + nA\right)}{\alpha nA + \frac{r-a}{b}} - e \end{bmatrix} \tag{11}$$

We observe that one eigenvalue is $\lambda_1 = -r+a$ and the other eigenvalue is $\lambda_2 = \dfrac{c\gamma\left(\frac{r-a}{b} + nA\right)}{\alpha nA + \frac{r-a}{b}} - e$. Based on these eigenvalues, then we have the following theorem.

Theorem 4. *The equilibrium point E_2 is locally asymptotically stable if and only if $r > a$ and $\alpha nA + \dfrac{r-a}{b} < e$.*

This equilibrium represents prey-only persistence. It occurs when the prey population can grow sufficiently despite natural mortality, while the predator cannot survive due to either inadequate predation opportunities or insufficient food supplementation.

The Jacobian matrix J_3 at E_3 is given by

$$J(E_3) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}. \tag{12}$$

with the coefficients given by:

$$a_{11} = \frac{r}{1+fy} - a - 2bx^* - \frac{cy^*(my^*+\alpha nA)}{(my^*+\alpha nA+x^*)^2}, \qquad a_{12} = -\frac{rfx^*}{(1+fy^*)^2} - \frac{cx^*(my^*+\alpha nA+x^*)-cmx^*y^*}{(my^*+\alpha nA+x^*)^2}, \tag{13}$$

$$a_{21} = \frac{cy^*\gamma(my^*+\alpha nA+x^*)-cy^*\gamma(x^*+nA)}{(my^*+\alpha nA+x^*)^2}, \qquad a_{22} = \frac{c\gamma(x^*+nA)(\alpha nA+x^*)}{(my^*+\alpha nA+x^*)^2} - e. \tag{14}$$

Due to the complicated nonlinear expressions in J_3, the eigenvalues cannot be expressed explicitly. However, by computing the trace and determinant of the Jacobian matrix, we can determine the local stability of E_3.

Theorem 5. *The interior equilibrium point $E_3 = (x^*, y^*)$ is locally asymptotically stable if and only if:*

1. *The trace of the Jacobian matrix is negative,* $\mathrm{Tr}(J(E_3)) = a_{11} + a_{22} < 0$,
2. *The determinant of the Jacobian matrix is positive,* $\det(J(E_3)) = a_{11}a_{22} - a_{12}a_{21} > 0$.

Otherwise, if $\mathrm{Tr}(J(E_3)) > 0$ *or* $\det(J(E_3)) < 0$, *then the equilibrium point E_3 is unstable.*

This interior equilibrium reflects coexistence between prey and predator. Its existence and stability depend on a delicate balance between prey growth, fear-induced reduction in foraging, predator feeding efficiency, and the availability of additional food. The Jacobian conditions ensure resilience to small perturbations around this steady state.

4. Numerical Results

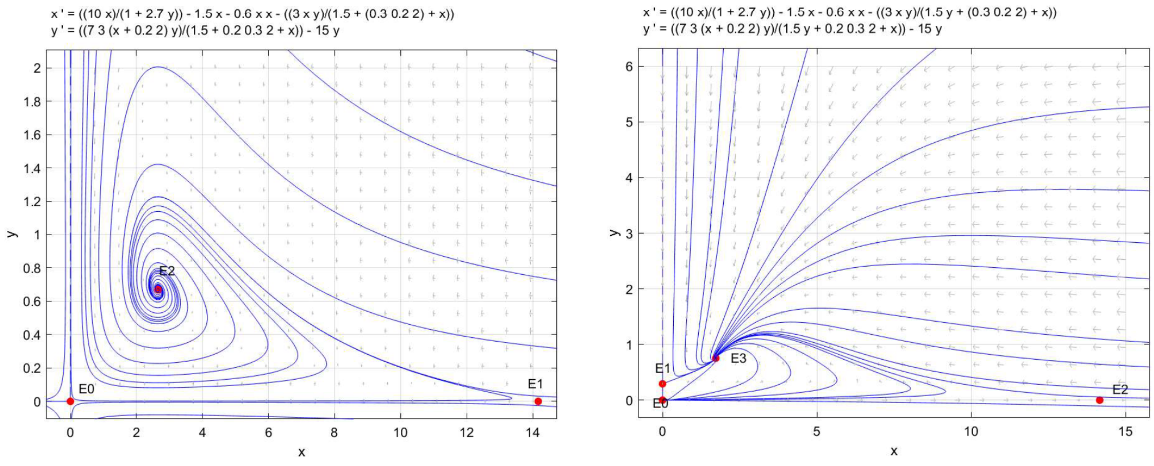
In this section, we numerically investigate the dynamic behavior of the model (6), based on the analytical findings discussed in the previous section. The parameter values used in the simulations are listed in Table 2.

Table 2. Parameter values used in the numerical simulation

Parameter	Value	Reference
r	10	Mondal et al. (2018)
a	1.5	Mondal et al. (2018)
f	2.7*	Mondal et al. (2018)
b	0.6	Mondal et al. (2018)
c	3	Mondal et al. (2018)
m	1.5	Mondal et al. (2018)
γ	7	Mondal et al. (2018)
e	15	Mondal et al. (2018)
A	2*	Mondal et al. (2018)
α	0.3	Mondal et al. (2018)
n	0.2	Mondal et al. (2018)

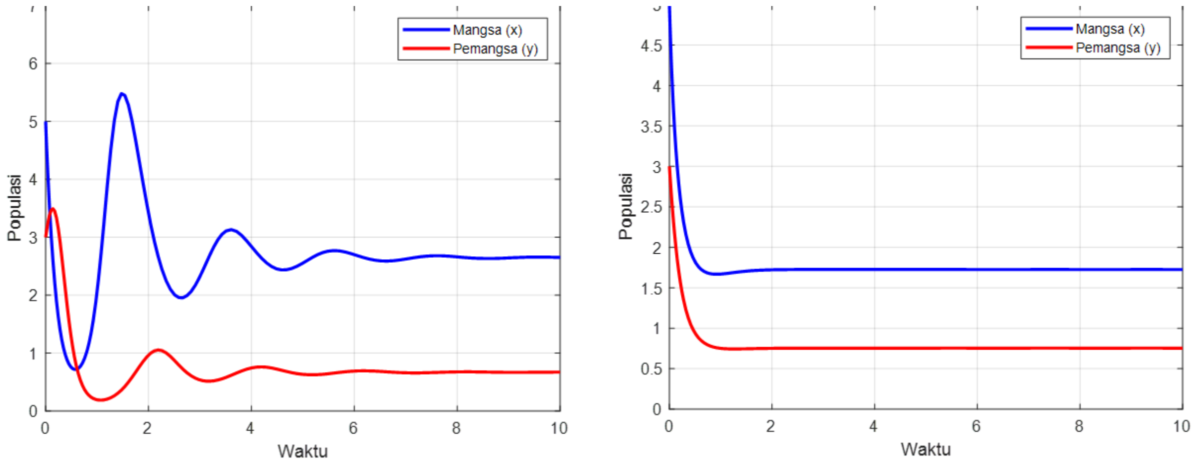
*Parameters f and A are varied in continuation simulations to analyze the behavior of equilibrium points.

As illustrated in Figure 1, the phase portraits and time series plots reveal clear and significant differences in the dynamic behavior of the system under the two types of functional responses. In the case of the Holling Type II response (Figure 1a), the system exhibits three equilibrium points. The origin $E_0 = (0,0)$ is unstable, indicating that total extinction of both species is not a stable long-term outcome. A boundary equilibrium exists at $E_1 = (14.17, 0)$, which functions as a saddle point



(a) Phase portrait with Holling Type II [3]

(b) Phase portrait with ratio-dependent response



(c) Time series with Holling Type II [3]

(d) Time series with ratio-dependent response

Figure 1. Comparison between Holling Type II and ratio-dependent functional responses based on phase portraits and time series simulations

and reflects a biologically unrealistic scenario in which the prey population survives in the complete absence of predators. The most biologically relevant equilibrium is the interior point $E_2 = (2.5, 0.67)$, which is locally asymptotically stable and indicates a state of sustainable coexistence between prey and predator populations. However, the system's trajectory toward this equilibrium is marked by high-amplitude oscillations, as illustrated in the time series plot (Figure 1c). These oscillations imply that the populations undergo several cycles of rapid growth followed by abrupt decline before stabilizing. Such patterns may not be favorable for long-term ecological stability, as the extreme fluctuations increase the risk of stochastic extinction, particularly when population densities fall to very low levels.

In contrast, the dynamics under the ratio-dependent functional response (Figure 1b) exhibit a more stable and smooth convergence toward equilibrium. The system has four equilibrium points: the origin, two saddle points at $(0, 0.29)$ and $(14.17, 0)$, and a stable interior equilibrium at $E_3 = (1.72, 0.75)$. The trajectories in the phase portrait approach this interior equilibrium more directly and with minimal oscillations, reflecting a rapid and steady stabilization process. This is further confirmed by the time series plot in Figure 1d, in which both prey and predator populations converge monotonically toward their equilibrium values. From a biological perspective, the ratio-dependent model represents a more resilient and realistic ecological interaction. By accounting for predator interference and the ratio of prey to predators, this functional response captures the natural limitations experienced by predators in crowded environments, resulting in more stable and ecologically robust population dynamics.

To further investigate the influence of key biological factors, we perform a bifurcation analysis focusing on two parameters: the anti-predator behavior parameter f, which modulates the strength of prey avoidance, and the supplemental feeding parameter A, which provides additional resources for predators independent of the prey population.

As shown in Figure 2, the system undergoes a transcritical bifurcation at the critical threshold $f = 7.69$, marking a significant shift in the nature and stability of its equilibrium points. Prior to this bifurcation—for instance, at $f = 2.7$—the dynamics converge to a stable interior equilibrium (Figure 3(a), (b)), representing a biologically favorable state where both prey and predator populations coexist.

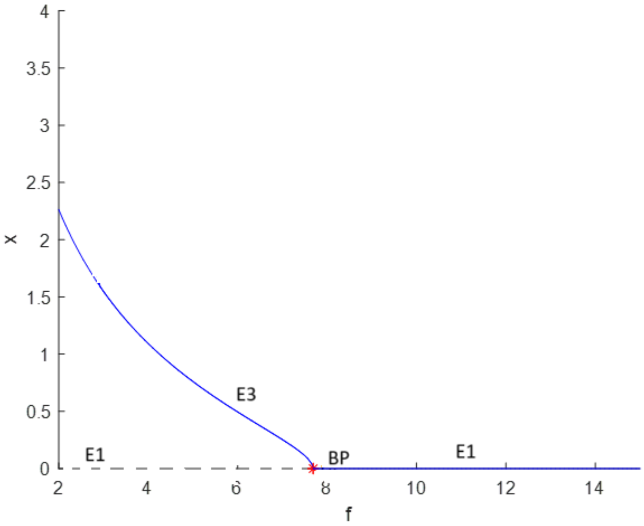


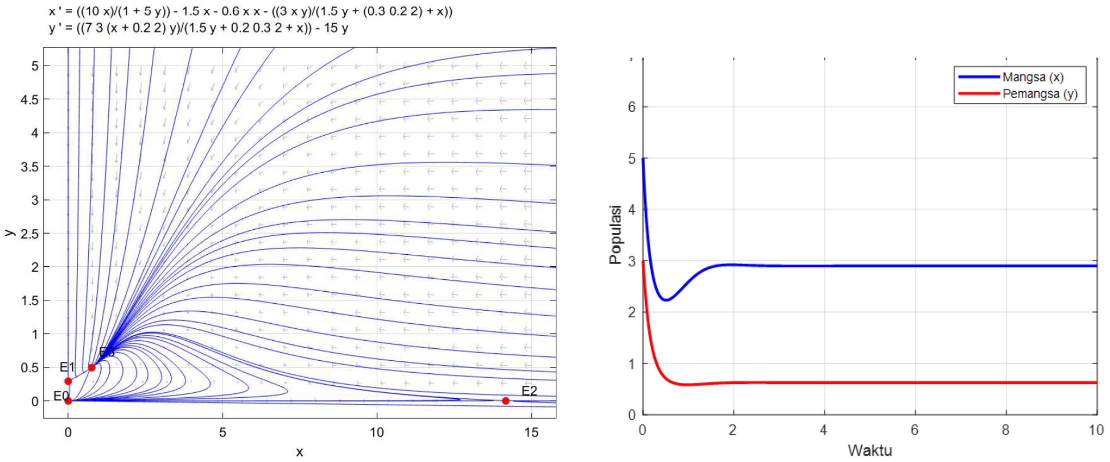
Figure 2. Bifurcation diagram showing a transcritical bifurcation at $f = 7.69$

As the parameter f increases, reflecting a stronger fear effect on the prey population, the system gradually approaches a critical regime. At the bifurcation point $f = 7.69$ (Figure 3(c), (d)), the interior equilibrium loses its stability, and a boundary equilibrium—where only predators persist—emerges as the new attractor. Beyond this threshold, for example at $f = 10$, the system settles into this predator-only equilibrium (Figure 3(e), (f)), resulting in the extinction of the prey population.

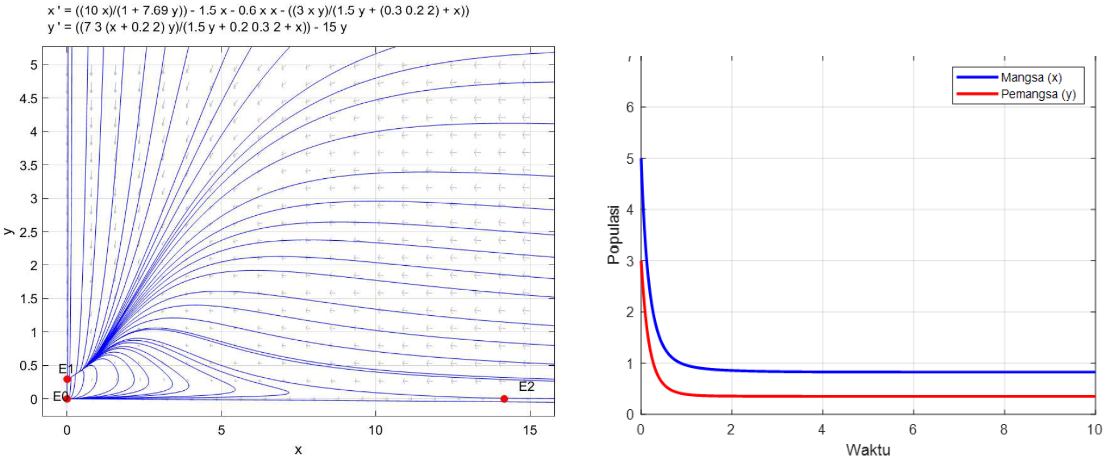
From a biological perspective, this transition illustrates the trade-off faced by prey species between predator avoidance and sustaining essential activities such as foraging and reproduction. An intensifying fear effect may lead prey to limit their exposure to predators, but at the cost of reducing their energy intake and reproductive output. Consequently, prey populations may decline even in the absence of direct predation, while predator populations remain viable due to supplemental feeding.

This behavior highlights the system's sensitivity to changes in the parameter f. Small variations around the critical value can cause qualitative shifts in the long-term dynamics. This finding underscores the ecological vulnerability of systems influenced by behaviorally-mediated interactions, especially under environmental pressures that heighten fear or stress in prey—such as habitat fragmentation, heightened predation risk, or anthropogenic disturbances.

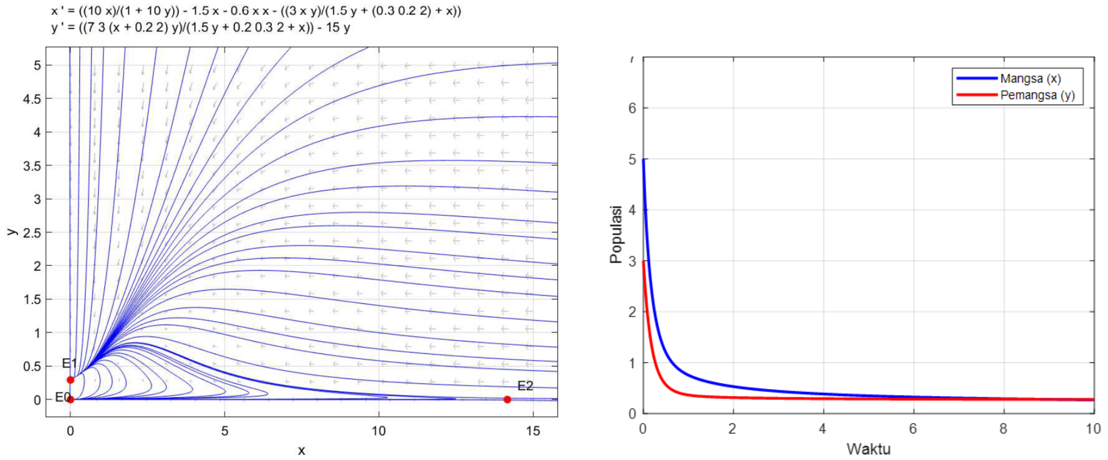
A comparable bifurcation pattern emerges when the parameter associated with supplemental feeding, denoted by A, is varied. As illustrated in Figure 4, the system exhibits a transcritical bifurcation at the critical value $A = 5.6965$. For values of A below this threshold, the predator population



(a) $f = 2.7$ **(b)** $f = 2.7$



(c) $f = 7.69$ **(d)** $f = 7.69$



(e) $f = 10$ **(f)** $f = 10$

Figure 3. Phase portraits and time series illustrating the system dynamics before, during, and after the transcritical bifurcation at $f = 7.69$

remains highly dependent on the natural prey for survival, resulting in a stable interior equilibrium where both species are able to coexist in the long term. However, as A increases and exceeds this criti-

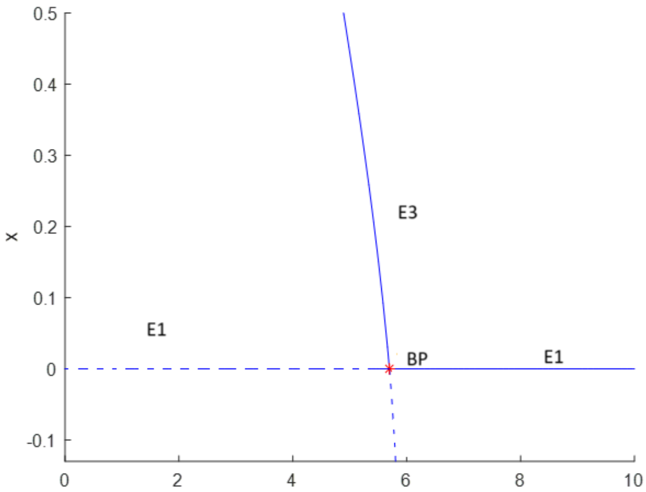


Figure 4. Bifurcation diagram illustrating a transcritical bifurcation occurring at $A = 5.6965$

cal point, the predator becomes progressively less reliant on the prey due to the increasing availability of alternative food sources. This shift alters the system dynamics considerably, leading to a destabilization of the coexistence equilibrium. Eventually, the prey population declines toward extinction, while the predator population stabilizes at a positive level supported solely by the supplemental food. This transition signifies the emergence of a boundary equilibrium, in which the prey is absent and the predator persists independently.

These dynamic transitions are further illustrated in Figure 5, which presents phase portraits and time series for three representative scenarios: before the bifurcation point ($A = 2$), at the bifurcation point ($A = 5.6965$), and after the bifurcation ($A = 6.7$).

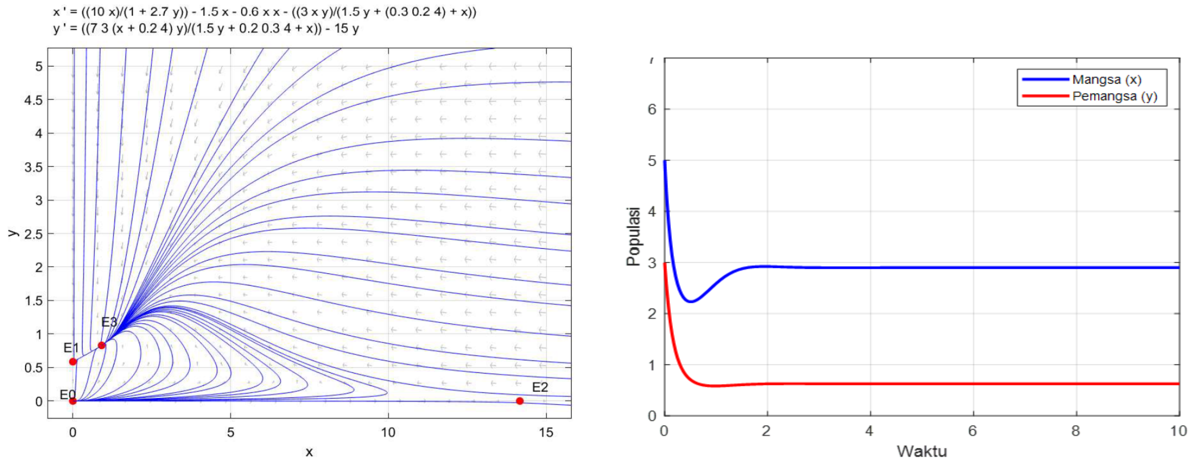
Prior to the bifurcation, the system tends toward a stable interior equilibrium where both predator and prey coexist. At the critical value, the interior and boundary equilibria coincide, indicating the presence of a transcritical bifurcation. Beyond this point, the system converges to a predator-only equilibrium as the prey population collapses.

Biologically, the parameter A represents the availability of supplemental food provided to the predator, independent of the prey population. When A is relatively low, predators rely heavily on capturing prey for sustenance, thereby maintaining a dynamic balance between both populations. In this regime, the predator cannot survive without the prey, and as such, the system converges to a stable interior equilibrium that supports coexistence.

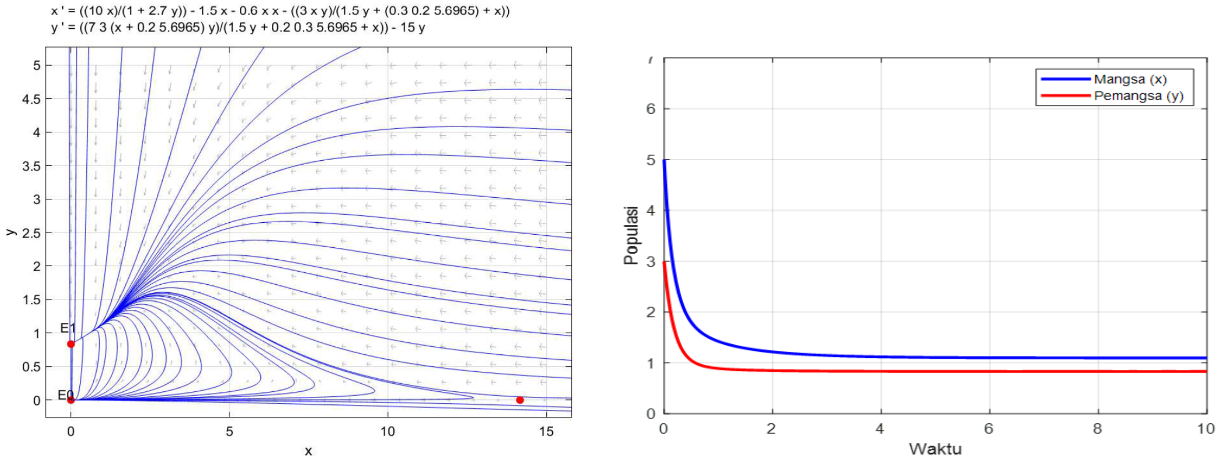
However, as A increases beyond the critical threshold, the predator becomes less dependent on the prey due to the growing availability of alternative food sources. This shift alters the system dynamics significantly: the prey population begins to decline, eventually reaching extinction, while the predator population stabilizes at a non-zero level sustained solely by the supplemental food. This scenario corresponds to the emergence of a boundary equilibrium, which replaces the interior equilibrium as the system's attractor.

Ecologically, this transition raises concerns regarding interventions that provide artificial sustenance to predator populations—such as in conservation areas or controlled ecosystems—without considering the cascading impacts on prey species. The model demonstrates that an excessive supply of supplemental food can inadvertently destabilize predator-prey coexistence by allowing predators to persist regardless of prey density, ultimately driving prey to extinction.

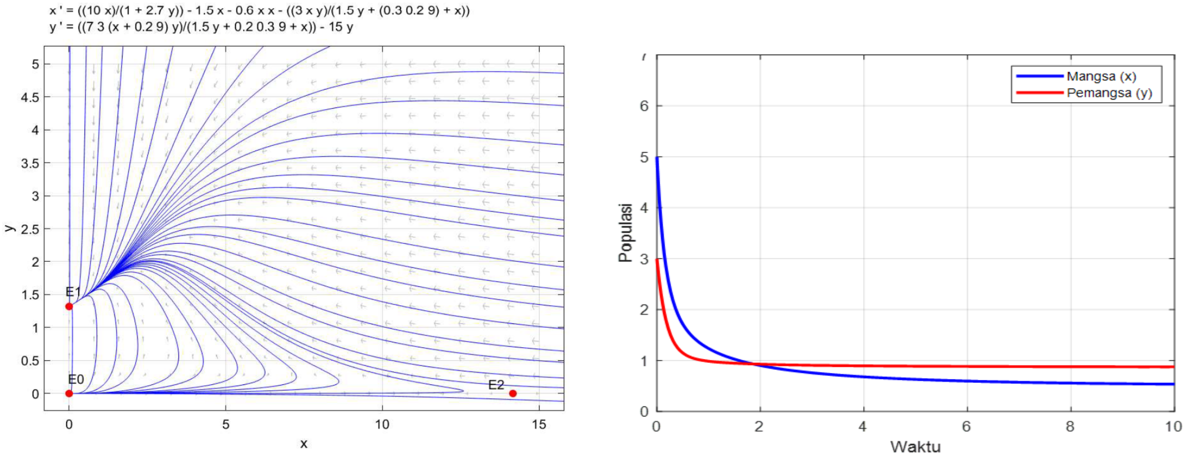
From a mathematical standpoint, this bifurcation once again emphasizes the system's sensitivity to key parameters. As with the fear effect parameter f, small variations in A near its critical value can trigger qualitative changes in the equilibrium structure and long-term outcomes. The presence



(a) Phase portrait: Before critical A **(b)** Time series: Before critical A



(c) Phase portrait: At critical A **(d)** Time series: At critical A



(e) Phase portrait: After critical A **(f)** Time series: After critical A

Figure 5. Phase portraits and time series illustrating the system dynamics before, during, and after the transcritical bifurcation at $A = 5.6965$

of a transcritical bifurcation in both parameter regimes—f and A—suggests a rich interplay between behavioral responses and environmental inputs, with profound implications for ecosystem stability and species survival.

5. Conclusion

This study examined a predator-prey model incorporating a ratio-dependent functional response, with behavioral and environmental modifications introduced via a fear parameter f and a supplemental food parameter A. The analysis revealed that both factors significantly alter the system's qualitative dynamics. Increased fear reduces prey foraging and reproduction, while elevated supplemental feeding enables predator survival even in prey-scarce environments. The system admits up to four biologically meaningful equilibria, with their existence and stability shaped by parameter interactions. Numerical simulations, including phase portraits and time series, demonstrate a range of ecological outcomes—coexistence, prey-only or predator-only persistence, and extinction of both species. Continuation analysis confirmed the occurrence of transcritical bifurcations as f and A cross critical thresholds, indicating abrupt structural changes in population dynamics. Nonetheless, the model assumes spatial homogeneity and deterministic dynamics, thereby neglecting the influence of spatial heterogeneity, stochastic environmental fluctuations, and delayed responses. Future research could extend this framework by incorporating stochasticity, time delays, or spatial structures, such as diffusion or patch-based dynamics. In conclusion, this work highlights the delicate balance in predator-prey systems driven by behavioral and environmental factors, and underscores the importance of integrating both ecological interactions and adaptive responses to better understand population dynamics and ecosystem resilience.

Supplementary Information

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Conflict of interest. The author declares that he has no conficts of interest to report regarding the present study.

Competing interests. The authors declare no competing interests.

Data availability. This study is based on parameter values adapted from Mondal et al. [3]. The authors also performed simulations by varying the fear parameter f and the additional food parameter A. All data are available from the corresponding author upon reasonable request.

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